Seminar in Polynomial Methods in Incidence Geometry, Harmonic Analysis and Number Theory

An incidence bound for lines in three dimensions

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1 Prerequisites

In this section we state, without proving, the main results used in the rest of this talk. Aside from the name of the result, in the bracket can be found the indexing number of the result as in [1].

Definition 1.1. (Number of Incidences) Let $\mathcal{S}$ be a set of points and $\mathcal{L}$ a set of lines. We define the **number of incidences** as

$$I(\mathcal{S},\mathcal{L}) = |\{(p,l) \in \mathcal{S} \times \mathcal{L} : p \in l\}|$$

Definition 1.2. (Flat and Critical points, Section 11.5) A point x on a smooth surface in $\mathbb{R}^3$ is called a **flat point** if there is a plane that is tangent to the surface at x to second order.
A point on a surface given by a polynomial P is called **critical** if all the partial derivatives $\partial_i P$ in that point equal zero.
We say that a point $x \in Z(P)$ is **special** if x is critical or flat. We say that a line $l \subset Z(P)$ is **special** if each point of the line is special.

Theorem 1.1. *(Polynomial partitioning, Theorem 10.3) For any dimension n, we can choose $C(n)$ such that the following holds. If X is any finite subset of $\mathbb{R}^n$ and D is any degree, then there is a non-zero polynomial $P \in Poly_D(\mathbb{R}^n)$ such that $\mathbb{R}^n \backslash Z(P)$ is a disjoint union of $\lesssim D^n$ open sets O_i each containing $\leq C(N)|X|D^{-n}$ points of X.*

The previous theorem's proof is based on topological arguments. The proof given in [1] is based on the general ham sandwich theorem, proven by Stone and Tukey.

Theorem 1.2. *(General ham sandwich theorem) Let V be a vector space of continuous functions on $\mathbb{R}^n$. Let $U_1, \dots, U_N \subset \mathbb{R}^n$ be finite volume open sets with $N < \dim V$. For any function $f \in V\backslash\{0\}$, suppose that $Z(f)$ has Lebesgue measure 0. Then there exists a function $f \in V\backslash\{0\}$ which bisects each set U_i.*

For a function $f : \mathbb{R}^n \to \mathbb{R}$ we say that bisects a finite volume open set U if

$$\text{vol}_n\{x \in U : f(x) > 0\} = \text{vol}_n\{x \in U : f(x) < 0\} = \frac{1}{2}\text{vol}_n U$$

The proof of the ham sandwich theorem is based on the Borsuk-Ulam theorem.

Theorem 1.3. *(Borsuk-Ulam) Suppose $\Phi : S^N \to \mathbb{R}^N$ is a continuous map that obeys the antipodal condition $\Phi(-x) = -\Phi(x)$ for all $x \in S^N$. Then the image of Φ contains 0.*

The ham sandwich theorem is proven by defining functions

$$\Phi_i(F) := \mathrm{vol}(\{x \in U_i : F(x) > 0\}) - \mathrm{vol}(\{x \in U_i : F(x) < 0\})$$

And assembling $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_N)$. Then we use the Borsuk-Ulam theorem to find a function F such that $\Phi(F) = 0$. Notice that it is equivalent to F bisecting all of the sets U_i. The next step in proving the polynomial partitioning theorem is making a version of the ham sandwich theorem for polynomials where bisecting a finite set S of points means the same number of points are on both sides of the curve given by the polynomial. Finally, we use the polynomial ham sandwich theorem repeatedly to prove the polynomial partitioning theorem.

Lemma 1.1. *(Plane detection lemma, Section 11.5) For any polynomial $P \in \mathbb{R}[x_1, x_2, x_3]$, we can associate a list of polynomials SP with the following properties:*

1. *If $x \in Z(P)$ then $SP(x) = 0$ iff x is critical or flat*
2. *If x is contained in three lines in $Z(P)$, then $SP(x) = 0$*
3. $\deg SP \leq 3 \deg P$
4. *If P is irreducible and SP vanishes on $Z(P)$ and $Z(P)$ contains a regular point, then $Z(P)$ is a plane.*

Theorem 1.4. *(Bezout, Theorem 6.7) Let $\mathbb{F}$ be an infinite field. If $P, Q \in Poly(\mathbb{F}^3)$ have no common factor, then the number of lines in $Z(P, Q) \subset \mathbb{F}^3$ is at most* $\deg P \cdot \deg Q$.

Bezout's theorem is one of the fundamental results of intersection theory. It has various applications. A more general version of the theorem can be found in William Fulton's book *Intersection Theory* in section 12.3. One of the consequences of the theorem is the Cayley-Bacharach theorem which we will state with two applications.

Theorem 1.5. *(Cayley-Bacharach) Let $X = Z(P_x)$ and $Y = Z(P_y)$ be two projective cubic curves over an algebraically closed field intersecting in 9 distinct points. If P is a cubic polynomial with zeroes in 8 out of 9 intersections, then $Z(P)$ passes through the last intersection.*

A simple proof of Bezout's theorem with applications can be found in book *Algebraic Curves* by Kazaryan, Lando and Prasolov. The first application of the Cayley-Bacharach theorem is Pappus' theorem.

Theorem 1.6. *(Pappus) Let l_1, l_2 be two lines and let $A_1, A_2, A_3 \in l_1 \backslash l_2$ and $B_1, B_2, B_3 \in l_2 \backslash l_1$ be disticnt points. Furthermore, let*

$$P_1 \in A_1B_2 \cap A_2B_1$$

$$P_2 \in A_2B_3 \cap A_3B_2$$

$$P_3 \in A_1B_3 \cap A_3B_1$$

Then, P_1, P_2 and P_3 are collinear.

Another application is proving the associativity of addition on elliptic curves. A detailed exposition can be found in *Rational Points on Elliptic Curves* by Silverman and Tate.

2 Motivation

We begin by mentioning the Szemerédi-Trotter theorem. The said theorem is a bound on the number of incidences of S points and L lines in $\mathbb{R}^2$.

Theorem 2.1. *(Szemerédi-Trotter) If $\mathcal{S}$ is a set of S points and $\mathcal{L}$ is a set of L lines in $\mathbb{R}^2$, then the number of incidences obeys the bound:*

$$I(\mathcal{S},\mathcal{L}) \lesssim S^{\frac{2}{3}}L^{\frac{2}{3}} + S + L$$

Since it is not the topic of this lecture, we give only a short sketch of the proof. The proof uses the following lemma and the polynomial partitioning theorem.

Lemma 2.1. *With $\mathcal{S}$ and $\mathcal{L}$ as above, then*

- $I(\mathcal{S},\mathcal{L}) \leq L + S^2$
- $I(\mathcal{S},\mathcal{L}) \leq L^2 + S$

Proof. Fix some point $x \in \mathcal{S}$. Let

$$L_x = \text{card}\{l \in \mathcal{L} : l \cap \mathcal{S} = \{x\}\}$$

For any other point y there exists at most one line in $\mathcal{L}$ containing both points. Now we see

$$I(x,\mathcal{L}) \leq S + L_x$$

By summation over all $x \in \mathcal{S}$ we get

$$I(\mathcal{S},\mathcal{L}) \leq S^2 + \sum_{x\in\mathcal{S}} L_x \leq S^2 + L$$

The other inequality can be obtained by fixing a line and summing over lines. □

Proof of Theorem 2.1. The proof goes by implementing the following steps:

Step I: If $L > \frac{S^2}{10}$ or $S > \frac{L^2}{10}$ then we reduce to the Lemma. So assume that $10^{\frac{1}{2}}S^{\frac{1}{2}} \leq L \leq \frac{S^2}{10}$.

Step II: We use induction on L.

Step III: We use polynomial partitioning theorem: we can find a non-zero polynomial P of degree $\leq D$ s.t. each component of the plane contains $\lesssim SD^{-2}$ points.

Step IV: We observe each component and apply the lemma.

Step V: We sum over cells.

Step VI: We obtain $I(\mathcal{S},\mathcal{L}) \leq C(LD + S^2D^{-2}) + I(\mathcal{S}_{alg},\mathcal{L}_{alg})$ Where sets with subscription alg denote intersections of the original sets with $Z(P)$.

Step VII: We apply the induction on the last term.

□

3 The analogue in three dimensions

The goal of this section is to prove the theorem that bounds the incidence number of lines and points in three dimensions. We begin by stating the theorem.

Theorem 3.1. *Let $\mathcal{S}$ be a set of S points and $\mathcal{L}$ a set of L lines in $\mathbb{R}^3$. Suppose that any plane contains no more that B lines of $\mathcal{L}$, and that $B \geq L^{\frac{1}{2}}$. There exists a constant C_0 independent of $\mathcal{L}$ and $\mathcal{S}$ such that*

$$I(\mathcal{S}, \mathcal{L}) \leq C_0 \left[S^{\frac{1}{2}} L^{\frac{3}{4}} + B^{\frac{1}{3}} L^{\frac{1}{3}} S^{\frac{2}{3}} + L + S \right]$$

The proof of this theorem follows in a similar manner as proof of Szemerédi-Trotter theorem. We do a polynomial cell decomposition with some polynomial P. Then we observe three different contributions of incidences:

- We use the polynomial partitioning theorem to control the incidences in the cells (outside of $Z(P)$.
- We bound planar parts of $Z(P)$ by knowing that each plane contains at most B lines.
- We bound non-planar parts of $Z(P)$ by the theory of flat points and lines.

Proof of Theorem 3.1: The proof uses induction on L. We assume that the statement holds for all sets of lines of cardinality $< L$. Notice that the argument used in the proof of Lemma 2.1 is independent of the dimension. Therefore we still have estimates

$$I(\mathcal{S}, \mathcal{L}) \leq S^2 + L$$

$$I(\mathcal{S}, \mathcal{L}) \leq S + L^2$$

The reader should consult proof of Proposition 8.1 in [1] to verify that the random projection argument combined with the Szemerédi-Trotter theorem yields an estimate

$$I(\mathcal{S}, \mathcal{L}) \lesssim \left[S^{\frac{2}{3}} L^{\frac{2}{3}} + L + S \right]$$

in any dimension. Let $D \geq 1$ be an integer to be chosen later. Now we apply the polynomial

partitioning theorem to obtain a polynomial $P \neq 0$ such that $\deg P \leq D$ and each component of $\mathbb{R}^3 \backslash Z(P)$ contains $\lesssim SD^{-3}$ points of $\mathcal{S}$. Now we define

$$\mathcal{S}_{alg} = \mathcal{S} \cap Z(P)$$

$$\mathcal{S}_{cell} = \mathcal{S} \backslash \mathcal{S}_{alg}$$

$$\mathcal{L}_{alg} = \mathcal{L} \cap Z(P)$$

$$\mathcal{L}_{cell} = \mathcal{L} \backslash \mathcal{L}_{alg}$$

Now we make the first estimate to separate algebraic parts.

Lemma 3.1. *(Cellular estimate) For some constant C,*

$$I(\mathcal{S}, \mathcal{L}) \leq C \left[D^{-\frac{1}{3}} L^{\frac{2}{3}} S^{\frac{2}{3}} + DL + S_{cell} \right] + I(\mathcal{S}_{alg}, \mathcal{L}_{alg})$$

To proceed with the proof we need to introduce new notation. Let $\mathcal{L}_{uniplan}$ be set of lines in $\mathcal{L}$ contained in exactly one plane in $Z(P)$ and $\mathcal{L}_{multiplan}$ be set of lines in $\mathcal{L}$ contained in more than one plane in $Z(P)$. Also set $\mathcal{L}_{plan} = \mathcal{L}_{uniplan} \cup \mathcal{L}_{multiplan}$. Let $\mathcal{S}_{uniplan}$ be set of points in $\mathcal{S}$ contained in exactly one plane in $Z(P)$ and $\mathcal{S}_{multiplan}$ be set of points in $\mathcal{S}$ contained in more than one plane in $Z(P)$. Also set $\mathcal{S}_{plan} = \mathcal{S}_{uniplan} \cup \mathcal{S}_{multiplan}$.

Now we notice that $Z(P)$ contains at most D planes, so we have

$$|\mathcal{L}_{multiplan}| \leq \binom{D}{2} \leq D^2$$

Now we continue to separate cases with the following lemma.

Lemma 3.2. *(Planar estimate)*

$$I(\mathcal{S}_{alg}, \mathcal{L}_{plan}) \leq C\left(B^{\frac{1}{3}} L^{\frac{1}{3}} S^{\frac{2}{3}} + DL + S_{uniplan}\right) + I(\mathcal{S}_{multplan}, \mathcal{L}_{multplan})$$

The previous lemma solves the case of planar parts of $Z(P)$. The following lemma uses the plane detection lemma to bound the the rest of the points on $Z(P)$. The following notation is needed for the lemma.

$$\mathcal{S}_{spec} = \{x \in \mathcal{S} : x \text{ special in } Z(P)\}$$

$$\mathcal{S}_{nonspec} = \mathcal{S} \backslash \mathcal{S}_{spec}$$

$$\mathcal{L}_{spec} = \{l \in \mathcal{L} : l \text{ special in } Z(P)\}$$

$$\mathcal{L}_{nonspec} = \mathcal{L} \backslash \mathcal{L}_{spec}$$

Lemma 3.3. *(Algebraic estimate)*

$$I(\mathcal{S}_{alg}, \mathcal{L}_{alg} \backslash \mathcal{L}_{plan}) \leq C(DL + S_{nonspec}) + I(\mathcal{S}_{spec}, \mathcal{L}_{spec} \backslash \mathcal{L}_{plan})$$

Proof. We observe

$$I(\mathcal{S}_{alg}, \mathcal{L}_{alg} \backslash \mathcal{L}_{plan}) \leq I(\mathcal{S}_{nonspec}, \mathcal{L}_{alg}) + I(\mathcal{S}_{spec}, \mathcal{L}_{nonspec}) + I(\mathcal{S}_{spec}, \mathcal{L}_{spec} \backslash \mathcal{L}_{plan})$$

Part (2) of the plane detection lemma implies that a point contained in three lines in $Z(P)$ must be special, so $I(\mathcal{S}_{nonspec}, \mathcal{L}_{alg}) \leq 2\mathcal{S}_{nonspec}$. Part (3) of the plane detection lemma states that $\deg SP \leq 3 \deg P$, hence $I(\mathcal{S}_{spec}, \mathcal{L}_{nonspec}) \leq 3DL$. ☐

It remains to estimate the contribution of the special lines. We separate the irreducible case and the general case.

Proposition 3.1. *(Irreducible case) If P is irreducible and $Z(P)$ is not a plane, then $Z(P)$ contains $\leq 3(\deg P)^2$ special lines,*

Proposition 3.2. *(General case) If P is any square-free non-zero polynomial, then there are at most $4(\deg P)^2$ special lines of $Z(P)$ that are not contained in any plane in $Z(P)$.*

We notice that this proposition exactly states that

$$|\mathcal{L}_{spec} \backslash \mathcal{L}_{plan}| \leq 4D^2$$

We restate inequality

$$|\mathcal{L}_{multiplan}| \leq D^2$$

Now we combine the three lemmas to obtain

$$I(\mathcal{S},\mathcal{L}) \le C\left[D^{-\frac{1}{3}}S^{\frac{2}{3}}L^{\frac{2}{3}} + DL + B^{\frac{1}{3}}L^{\frac{1}{3}}S^{\frac{2}{3}} + L + |\mathcal{S}\backslash(\mathcal{S}_{multiplan}\cup\mathcal{S}_{spec})|\right]$$

$$+I(\mathcal{S}_{multiplan}\cup\mathcal{S}_{spec},\mathcal{L}_{multiplan}\cup(\mathcal{L}_{spec}\backslash\mathcal{L}_{plan}))$$

Combining the two previously stated inequalities we obtain

$$|\mathcal{L}_{multiplan}\cup(\mathcal{L}_{spec}\backslash\mathcal{L}_{plan})| \le 10D^2$$

Now we choose D such that $1 \le D \le \frac{L^{\frac{1}{2}}}{10}$ to minimize the term in brackets above, and we will have bound

$$|\mathcal{L}_{multiplan}\cup(\mathcal{L}_{spec}\backslash\mathcal{L}_{plan})| \le \frac{L}{2}$$

so we can apply induction. Applying induction yields

$$I(\mathcal{S},\mathcal{L}) \le C\left[D^{-\frac{1}{3}}S^{\frac{2}{3}}L^{\frac{2}{3}} + DL + B^{\frac{1}{3}}L^{\frac{1}{3}}S^{\frac{2}{3}} + L + |\mathcal{S}\backslash(\mathcal{S}_{multiplan}\cup\mathcal{S}_{spec})|\right] +$$

$$C_0\left[S^{\frac{1}{2}}(L/2)^{\frac{3}{4}} + B\frac{1}{3}(L/2)^{\frac{1}{3}}S^{\frac{2}{3}} + \frac{L}{2} + |\mathcal{S}_{multiplan}\cup\mathcal{S}_{spec}|\right]$$

We choose C_0 sufficiently large compared to C. The rest of the proof is choosing $D \sim S^{\frac{1}{2}}L^{\frac{-1}{4}}$ to obtain

$$D^{\frac{-1}{3}}S^{\frac{2}{3}}L^{\frac{2}{3}} + DL \lesssim S^{\frac{1}{2}}L^{\frac{3}{4}} + B^{\frac{1}{3}}L^{\frac{1}{3}}S^{\frac{2}{3}}$$

We refer to counting lemma to assume $10L^{\frac{1}{2}} \le S \le \frac{1}{10}L^2 \implies 1 \le S^{\frac{1}{2}}L^{\frac{-1}{4}} \le L^{\frac{3}{4}}$. We choose $D = \min\{S^{\frac{1}{2}}L^{\frac{-1}{4}}, \frac{L^{\frac{1}{2}}}{10}\}$ and verify the rest by simple calculation. $\square$

It remains to prove the lemmas and propositions used.

Proof of Lemma 3.1: Denote by O_i components of $\mathbb{R}^3\backslash Z(P)$ and

$$\mathcal{S}_i = \mathcal{S}\cap O_i$$

$$\mathcal{L}_i = \{l \in \mathcal{L} : l \text{ intersects } O_i\}$$

Now we have

$$\sum S_i = S_{cell}$$

$$S_i \lesssim SD^{-3}$$

$$\sum L_i \lesssim DL$$

$$\implies I(\mathcal{S}_{cell},\mathcal{L}) = \sum_i I(\mathcal{S}_i,\mathcal{L}_i)$$

By Szemerédi-Trotter we have

$$\sum_i I(\mathcal{S}_i,\mathcal{L}_i) \lesssim \sum_i L_i^{\frac{2}{3}}S_i^{\frac{2}{3}} + L_i + S_i \lesssim DL + S_{cell} + \sum_i L_i^{\frac{2}{3}}S_i^{\frac{2}{3}}$$

We use $S_i \lesssim SD^{-3}$ and Hölder inequality to obtain

$$\sum_i L_i^{\frac{2}{3}}S_i^{\frac{2}{3}} \lesssim S^{\frac{1}{3}}D^{-1}\sum_i L_i^{\frac{2}{3}}S_i^{\frac{1}{3}} \le S^{\frac{1}{3}}D^{-\frac{1}{3}}(\sum_i L_i)^{\frac{2}{3}}(\sum_i S_i)^{\frac{1}{3}} \lesssim S^{\frac{1}{3}}D^{-1}(DL)^{\frac{2}{3}}S^{\frac{1}{3}} = D^{-\frac{1}{3}}S^{\frac{2}{3}}L^{\frac{2}{3}}$$

We combine those inequalities to get

$$I(\mathcal{S}_{cell}, \mathcal{L}) \lesssim D^{-\frac{1}{3}} S^{\frac{2}{3}} L^{\frac{2}{3}} + L + S_{cell}$$

We also know that each line passing through some cell contains at most D points from $\mathcal{S}_{alg}$. Therefore, we get

$$I(\mathcal{S}_{alg}, \mathcal{L}_{cell}) \leq DL$$

Now

$$I(\mathcal{S}_{cell}, \mathcal{L}) \leq I(\mathcal{S}_{cell}, \mathcal{L}) + I(\mathcal{S}_{alg}, \mathcal{L}_{cell}) + I(\mathcal{S}_{alg}, \mathcal{L}_{alg})$$

implies the statement of the lemma. □

Proof of Lemma 3.2: We just give a rough sketch of the proof of this lemma:

Step I: Observe $I(\mathcal{S}_{alg}, \mathcal{L}_{plan}) = I(\mathcal{S}_{alg}, \mathcal{L}_{uniplan}) + I(\mathcal{S}_{multiplan}, \mathcal{L}_{multiplan})$

Step II: It suffices to show $I(\mathcal{S}_{alg}, \mathcal{L}_{uniplan}) \leq B^{\frac{1}{3}} L^{\frac{1}{3}} S^{\frac{2}{3}} + DL + \mathcal{S}_{uniplan}$

Step III: Separate points and lines contained only in one plane from those contained in multiple planes. Bound incidences of points in multiple planes with lines in one plane by DL.

Step IV: Use Szemerédi-Trotter in each plane of $Z(P)$ and sum up the results.

Step V: Bound sum of $L_{uni\pi}^{\frac{2}{3}} S_{uni\pi}^{\frac{2}{3}}$ by using $L_{uni\pi} \leq B$ and Hölder inequality.

□

Proof of Proposition 3.1: Suppose that $Z(P)$ has a regular point. If SP vanished on $Z(P)$, part (4) of the plane detection lemma would imply $Z(P)$ was a plane. That contradicts the assumption. We conclude SP does not vanish on $Z(P)$. Suppose Q is one of the polynomials of SP not vanishing on $Z(P)$. We see that P and Q are coprime. Now we use our stated form of the Bezout theorem to see that $Z(P) \cap Z(Q)$ contains at most $3(\deg P)^2$ lines.

In case $Z(P)$ has no regular points, $\partial_i P$ vanishes on $Z(P)$ for each i. Now, since P is not constant $\exists i : \partial_i P \neq 0$. Since P is irreducible, P and $\partial_i P$ are coprime. By the Bezout theorem we have $Z(P) \subset Z(P) \cap Z(\partial_i P)$ contains at most D^2 lines.

□

Proof of Proposition 3.2. We assume $P = \prod P_j$ where P_j are irreducible and distinct. Let $l \subset Z(P_i), Z(P_j)$ be a line for $i \neq j$. Notice that $\forall x \in l, \quad \nabla P(x) = \sum_k (\nabla P_k) P_1 \dots P_{k-1} P_{k+1} \dots$ and each term vanishes at x (hence x is critical). Therefore, line l is special.

If $l \subset Z(P_j)$ for a unique j, we see that along l holds $\nabla P = (\nabla P_j) P_1 \dots P_{j-1} P_{j+1} \dots$. Since each other P_i vanishes only on finitely many points of l, we see that ∇P vanishes on l if and only if ∇P_j vanishes. We conclude l is a critical line of $Z(P_j)$ iff it is a critical line of $Z(P)$. In case l is not a critical line of $Z(P)$ or $Z(P_j)$, the only way for it to be special is that each of its regular points is flat. Flatness is a condition that can be checked on a small neighborhood $x \in U$. But there exists U such that $Z(P) \cap U = Z(P_j) \cap U$. So a regular point is flat for $Z(P)$ if and only if x is flat for $Z(P_j)$. We have shown that a line $l \subset Z(P)$ is special for P if and only if either l is special for some P_j or it lies in at least two $Z(P_j)$. Now we use the Bezout theorem to see that the number of lines lying in at least two $Z(P_j)$ is $\leq (\deg P)^2$. The number of special lines in each $Z(P_j)$ are bounded by the previous version of this proposition by $\leq 3(\deg P_j)^2$. So the total number of special lines in $Z(P)$ is bounded by

$$(\deg P)^2 + \sum_j 3(\deg P_j)^2 \leq 4(\deg P)^2$$

□

References

[1] Guth, Larry *Polynomial Methods in Combinatorics.* American Mathematical Society (2016)